

Control Systems I

Robustness

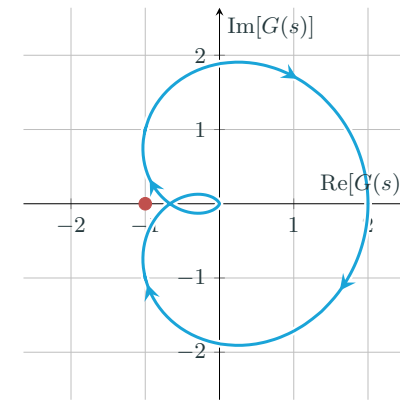
Colin Jones

Laboratoire d'Automatique

1

Uncertain Delay

Designed for system $G(s)$, but system in real-world $G'(s) = G(s)e^{-s}$

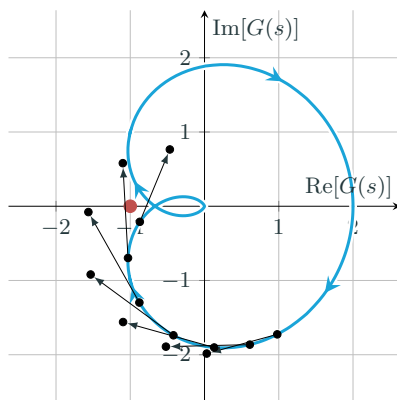


- Impact of delay : Phase delay of $-\omega$
- Original system is stable, delayed system is not

2

Uncertain Delay

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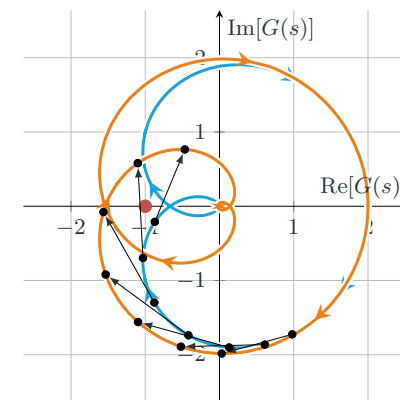


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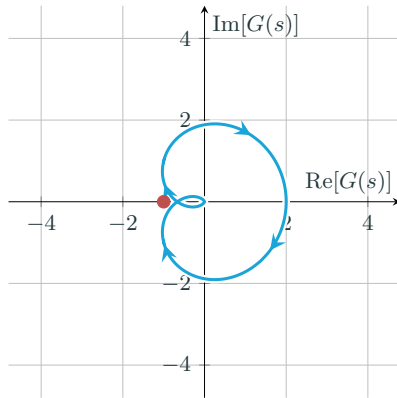


- Impact of delay : Phase delay of $-\omega$
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2

Example - Uncertain Gain

Design for system $G(s)$, but in real-world we get $G'(s) = \alpha G(s)$

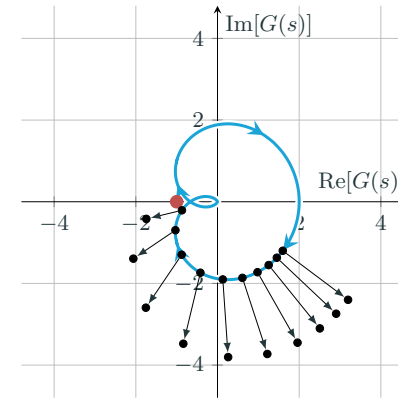


- Impact of uncertain gain : Scaling
- Original system is stable, system with larger gain is not

3

Example - Uncertain Gain

Design for system $G(s)$, but in real-world we get $G'(s) = \alpha G(s)$

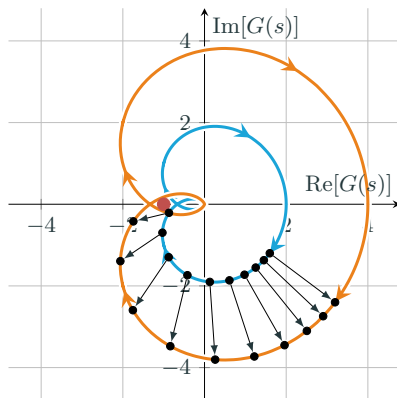


- Impact of uncertain gain : Scaling
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Example - Uncertain Gain

Design for system $G(s)$, but in real-world we get $G'(s) = \alpha G(s)$



- Impact of uncertain gain : Scaling
- Original system is stable, system with larger gain is not

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Robustness

The idea: The farther the 'nominal' Nyquist curve is from the -1 point, the more likely the real system will be stable.

The various margins measure distance in terms of some common terms:

- Uncertain gain
- Uncertain phase
- Uncertain delay

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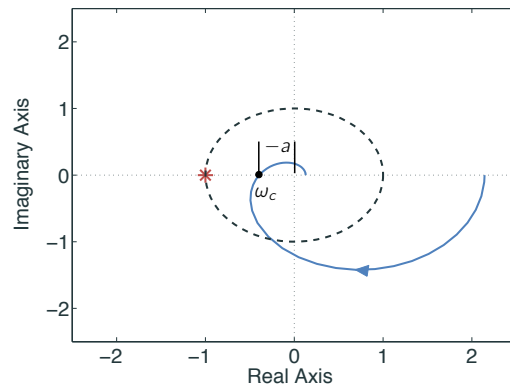
Gain Margin

The **gain margin** is the number $1/a$, where

$$a = \min_{\omega} K(j\omega)G(j\omega) \\ \text{s.t. } \text{Im } K(j\omega)G(j\omega) = 0$$

i.e., smallest negative crossing of the real axis

- Expressed in decibels
 $-20 \log_{10} a > 0$
- Amount that gain can increase while stable
- Between 4dB and 12dB generally considered safe
- $GM < 1$ (0dB) means that the closed-loop system is unstable, $GM > 1$ (0dB) that it is stable



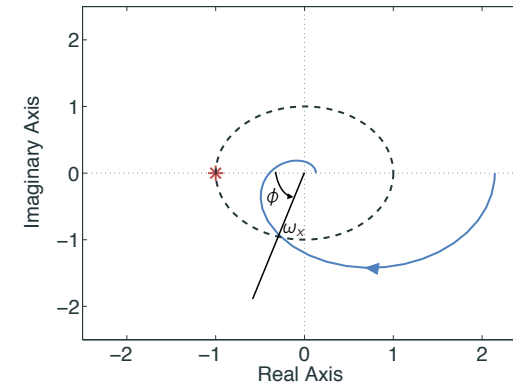
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Phase Margin

The **phase margin** ϕ is the smallest increase in phase that will cause instability

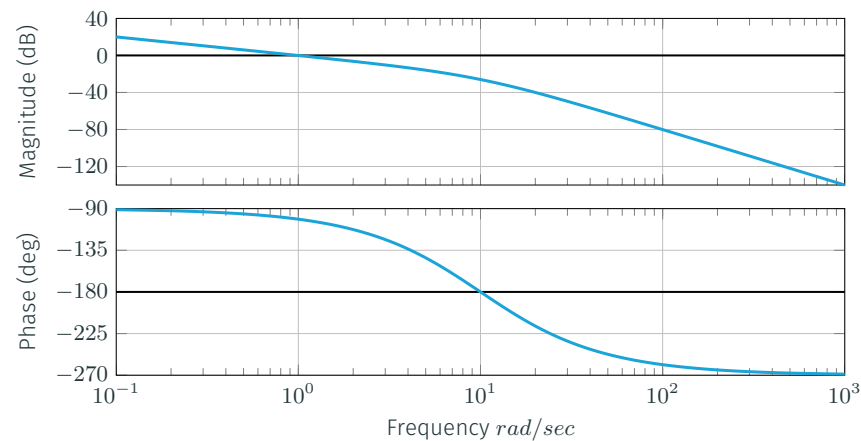
$$\phi = \min_{\phi, \omega} \phi \\ \text{s.t. } K(j\omega)G(j\omega)e^{j\phi} = -1$$

- Usually expressed in degrees
- Between 30° and 60° generally considered safe



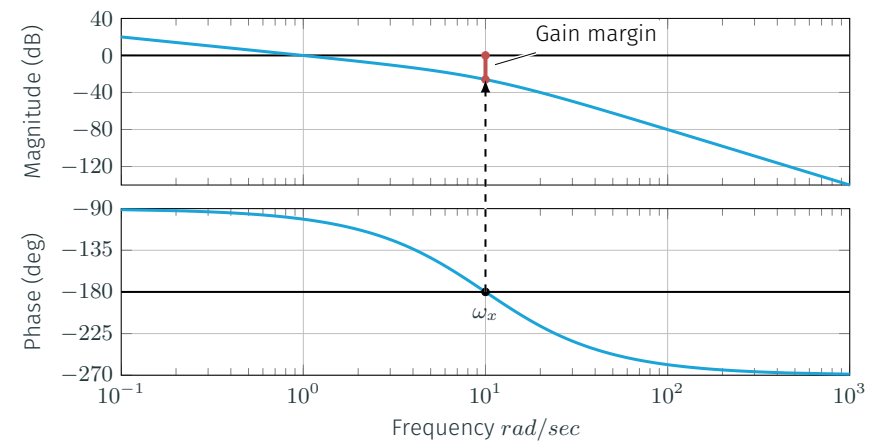
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Phase and Gain Margin on Bode Plot



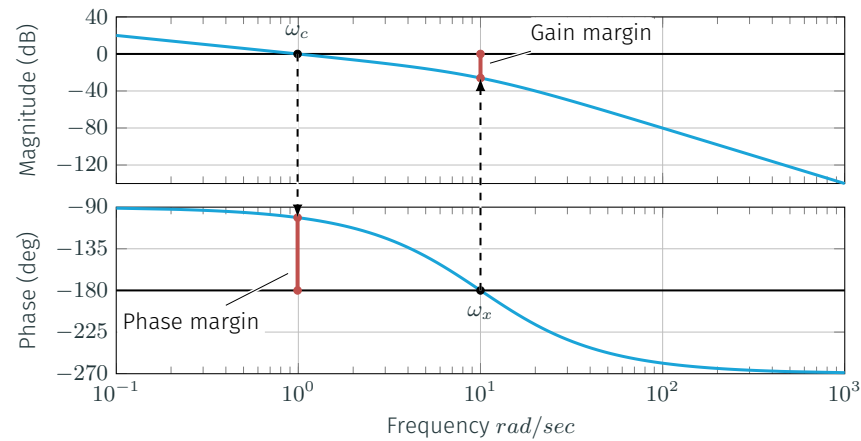
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Phase and Gain Margin on Bode Plot



7

Phase and Gain Margin on Bode Plot



7

Phase and Gain Margin on Bode Plot

Procedure:

1. Find frequency ω_c that system passes 0dB, from magnitude plot
2. Find frequency ω_x that system passes -180° , from the phase plot
3. Gain margin = $-20 \log_{10} |K(j\omega_x)G(j\omega_x)|$ dB
4. Phase margin = $\angle K(j\omega_c)G(j\omega_c) - 180$ in degrees

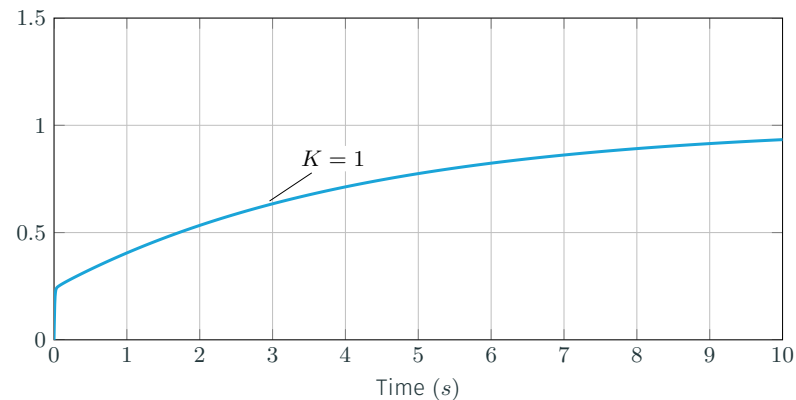
Gain and phase margin positive $\rightarrow$ stable

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Example: AFM

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s + 3000)(s + 1000)(s + 100)(s^2 + 50s + 6.25 \cdot 10^6)}$$

$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$

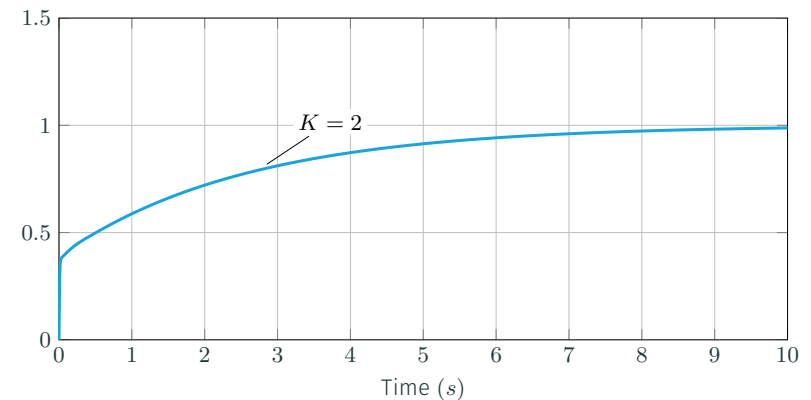


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Example: AFM

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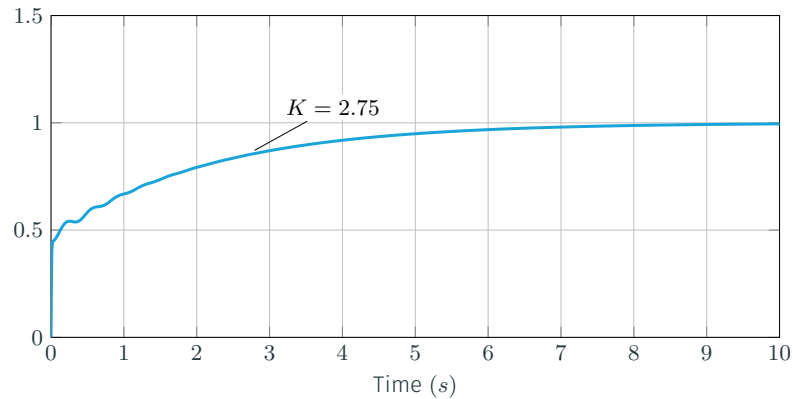


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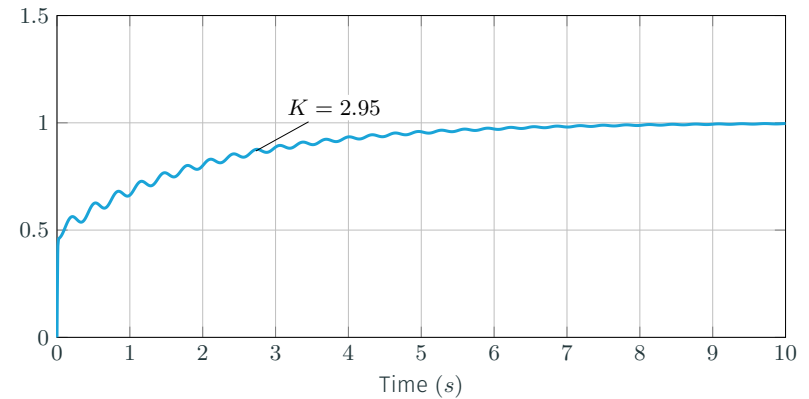


9

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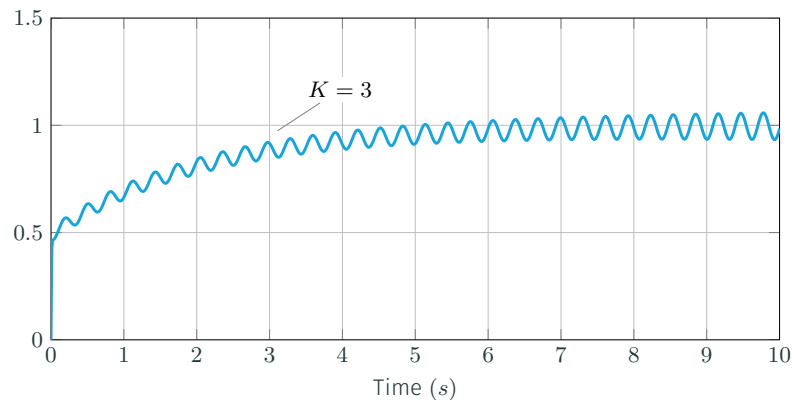


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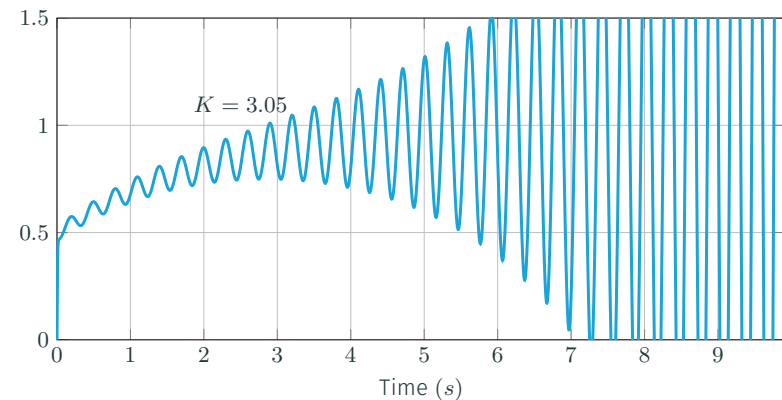


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Example: AFM

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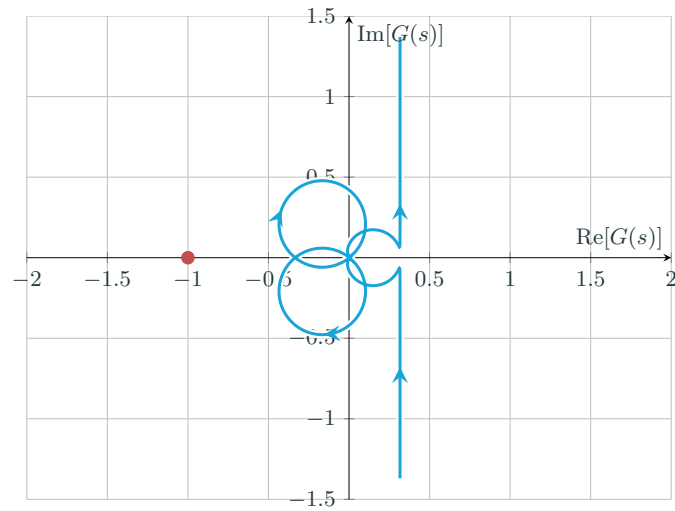
$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



System becomes unstable somewhere between $K = 3$ and $K = 3.05$

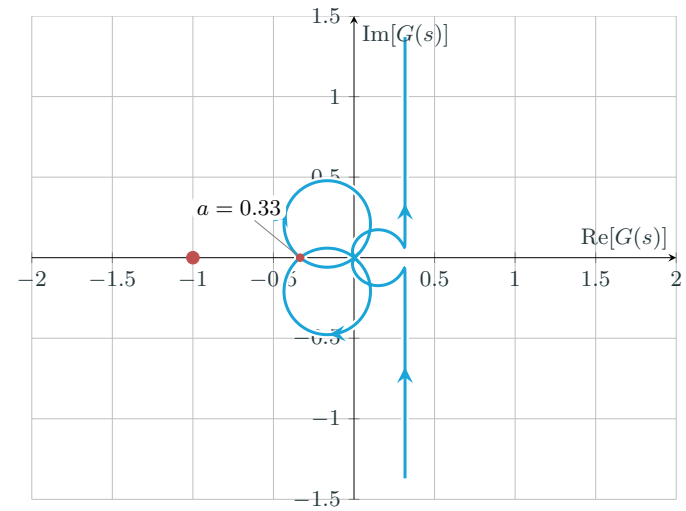
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Gain and Phase Margins : Nyquist



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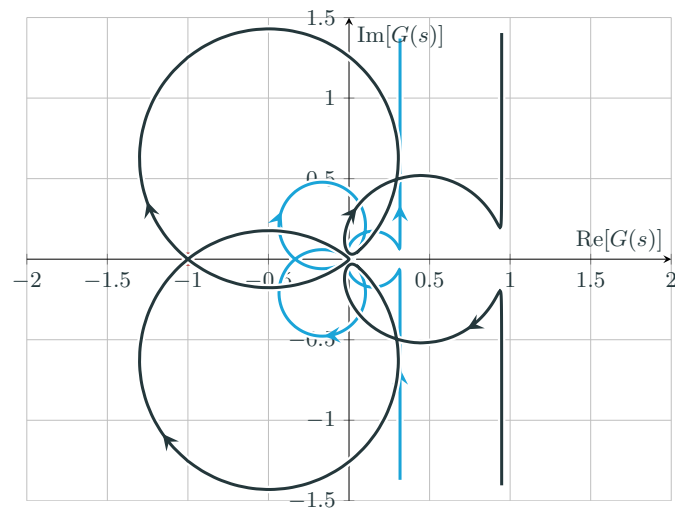
Gain and Phase Margins : Nyquist



Gain margin = $-20 \log_{10} a = 9.63 \text{ dB}$
System becomes unstable at a gain of $1/a = 3.03$

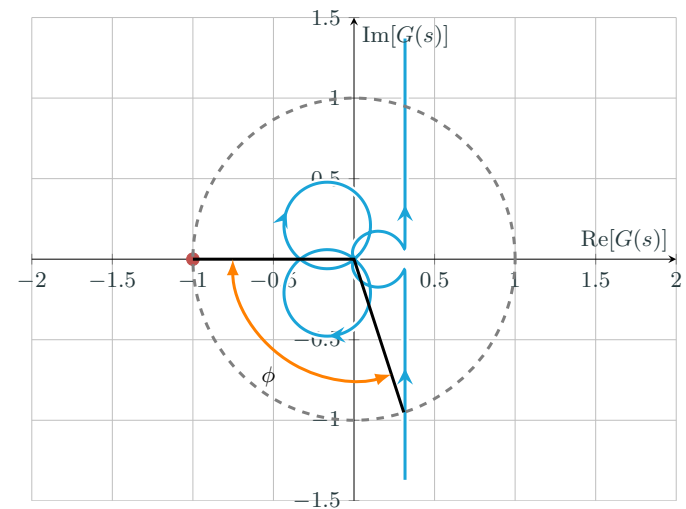
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Gain and Phase Margins : Nyquist



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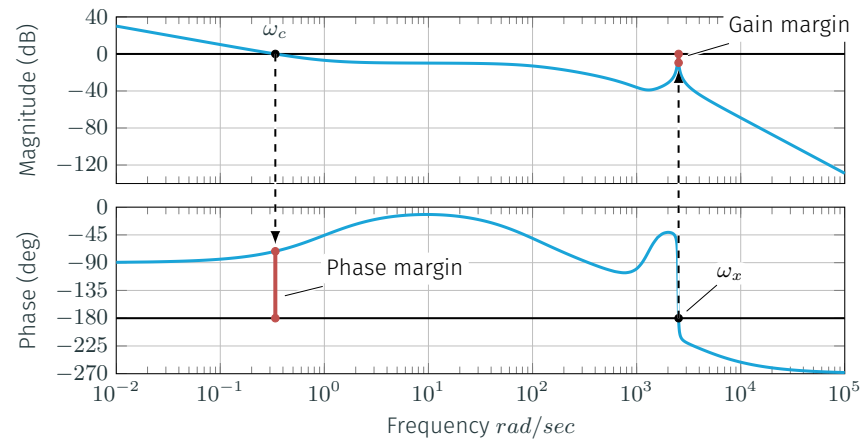
Gain and Phase Margins : Nyquist



Phase margin = 108 degrees

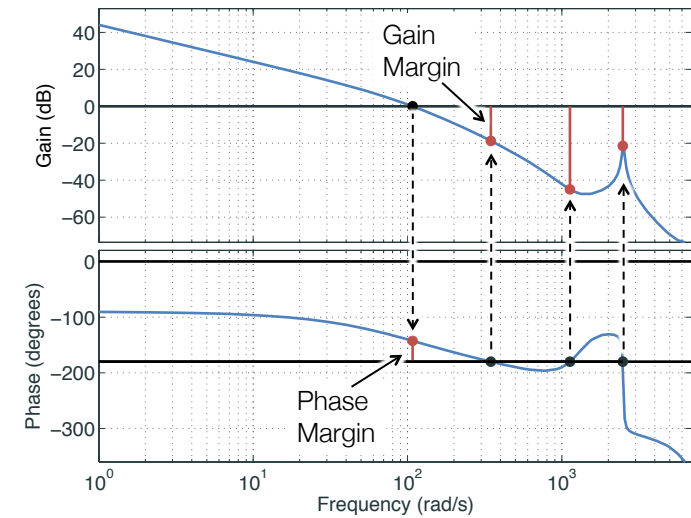
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Gain and Phase Margins : Bode



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Gain and Phase Margins : Bode



Choose the 'first to be unstable' if there are multiple crossings

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Margins - Summary

- Margins measure how far from stability the closed-loop system is in terms of a single uncertain parameter
 - Phase
 - Gain
 - Delay
- Many applications specify minimum phase and gain margins for safety
- In later lectures we will look at dynamic controllers that *shape* the frequency response so that we have good margins

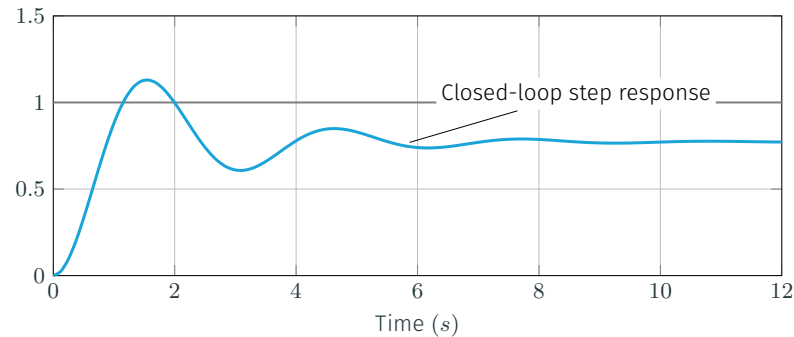
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Steady State Errors

Steady-State Offset

$$G(s) = \frac{3.4}{s^2 + s + 1}$$

$$K(s) = 1$$

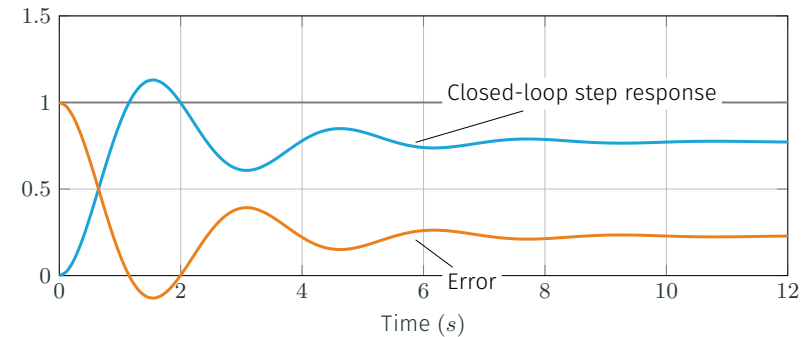


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Steady-State Offset

$$G(s) = \frac{3.4}{s^2 + s + 1}$$

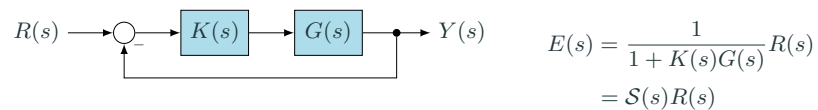
$$K(s) = 1$$



Stable does not mean that the output tracks the reference!

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Steady-State Offset



Theorem

Final Value Theorem

$$\lim_{t \rightarrow \infty} w(t) = \lim_{s \rightarrow 0} sW(s)$$

If poles of $sW(s)$ are in the left half plane

If $r(t) = 1, t \geq 0$, then $R(s) = 1/s$

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{1}{1 + K(s)G(s)} \cdot \frac{1}{s} = \lim_{s \rightarrow 0} \frac{1}{1 + K(s)G(s)} = ???$$

- Not infinite because $K(s)G(s) \neq -1$
- Called this the **steady-state offset** or **steady-state error**
- Conditions for the steady-state offset to be zero?

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System Type and Open-Loop Steady-State Gain

Suppose that the open-loop transfer function has q poles at $s = 0$

$$K(s)G(s) = \frac{B(s)}{s^q A(s)}$$

where A and B are polynomials.

The number q is called the **type** or the **class** of the open-loop system.

The **open-loop steady-state gain** of the system $K(s)G(s)$ is

$$\gamma := \lim_{s \rightarrow 0} s^q K(s)G(s) = \frac{B(0)}{A(0)}$$

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Type 0 System Response to a Step Command

Suppose we have a type 0 system

$$K(s)G(s) = \frac{B(s)}{A(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

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Type 0 System Response to a Step Command

Suppose we have a type 0 system

$$K(s)G(s) = \frac{B(s)}{A(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

The steady-state error for the closed-loop system will be

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} s \frac{1}{1 + K(s)G(s)} \cdot \frac{1}{s} \\ &= \frac{1}{1 + \gamma}\end{aligned}$$

17

Type 1 System Response to a Step Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

18

Type 1 System Response to a Step Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

The steady-state error for the closed-loop system will be

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} s \frac{1}{1 + K(s)G(s)} \cdot \frac{1}{s} \\ &= \lim_{s \rightarrow 0} \frac{s}{s + \frac{B(s)}{A(s)}} \\ &= 0\end{aligned}$$

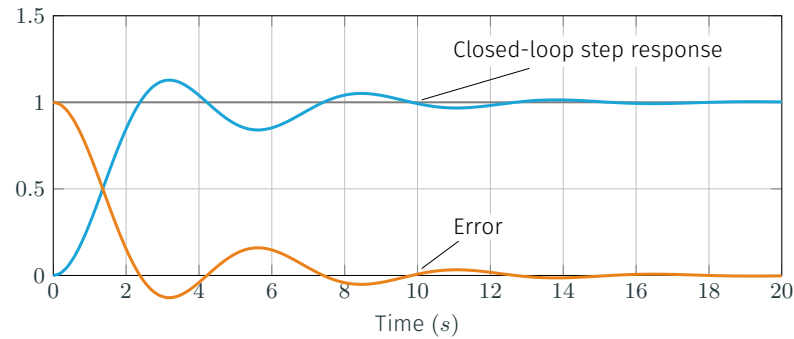
The steady-state error will be zero for all step inputs and systems!

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Type 1 System Response to a Step Command

$$G(s) = \frac{3.4}{s^2 + s + 1}$$

$$K(s) = \frac{1}{5} \left(1 + \frac{1}{s} \right)$$



Zero steady-state offset

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Type 1 System Response to a Ramp Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a **ramp input** $r(t) = t$

$$R(s) = \frac{1}{s^2}$$

20

Type 1 System Response to a Ramp Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a **ramp input** $r(t) = t$

$$R(s) = \frac{1}{s^2}$$

The steady-state error for the closed-loop system will be

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} s \cdot \frac{1}{1 + K(s)G(s)} \cdot \frac{1}{s^2} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + \frac{B(s)}{sA(s)}} \cdot \frac{1}{s} \\ &= \frac{1}{\gamma} \end{aligned}$$

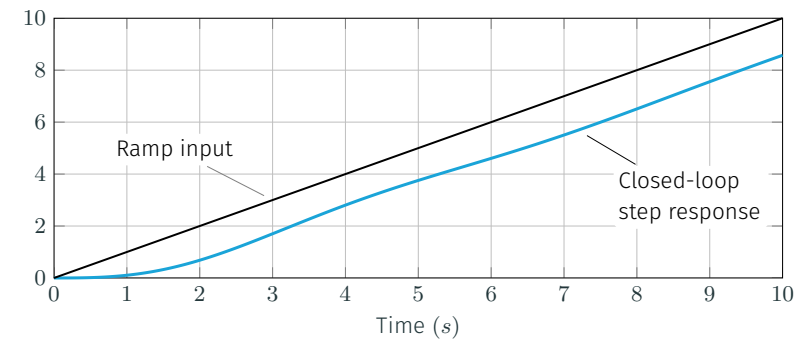
Non-zero steady-state error.

20

Type 1 System Response to a Ramp Command

$$G(s) = \frac{3.4}{s^2 + s + 1}$$

$$K(s) = \frac{1}{5} \left(1 + \frac{1}{s} \right)$$



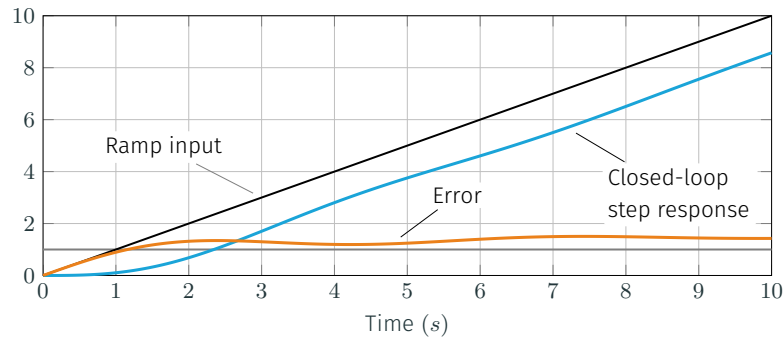
Constant steady-state offset

21

Type 1 System Response to a Ramp Command

$$G(s) = \frac{3.4}{s^2 + s + 1}$$

$$K(s) = \frac{1}{5} \left(1 + \frac{1}{s} \right)$$



Constant steady-state offset

21

Type 1 System Response to a Parabolic Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a *parabolic input* $r(t) = t^2$

$$R(s) = \frac{2}{s^3}$$

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Type 1 System Response to a Parabolic Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a *parabolic input* $r(t) = t^2$

$$R(s) = \frac{2}{s^3}$$

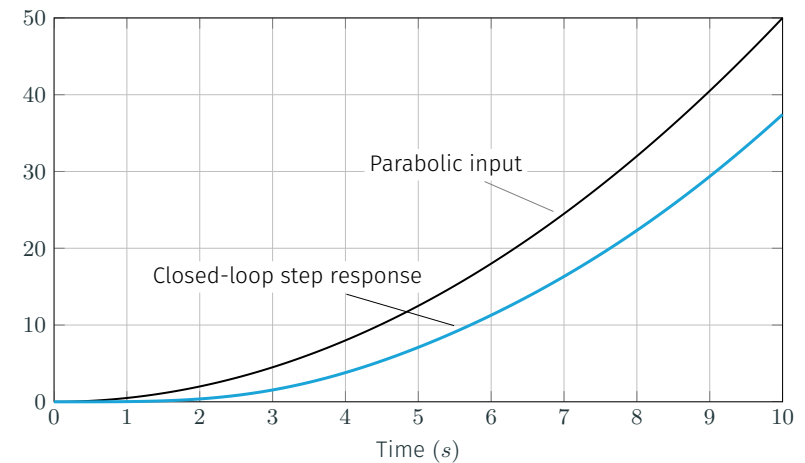
The error is

$$\begin{aligned} E(s) &= \frac{1}{1 + K(s)G(s)} \cdot \frac{2}{s^3} \\ &= \frac{sB(s)}{sA(s) + B(s)} \cdot \frac{2}{s^3} \\ &= \frac{B(s)}{sA(s) + B(s)} \cdot \frac{2}{s^2} \end{aligned}$$

Cannot apply final-value theorem because there is more than one pole at 0
→ Implies that the error is either unbounded, or oscillates

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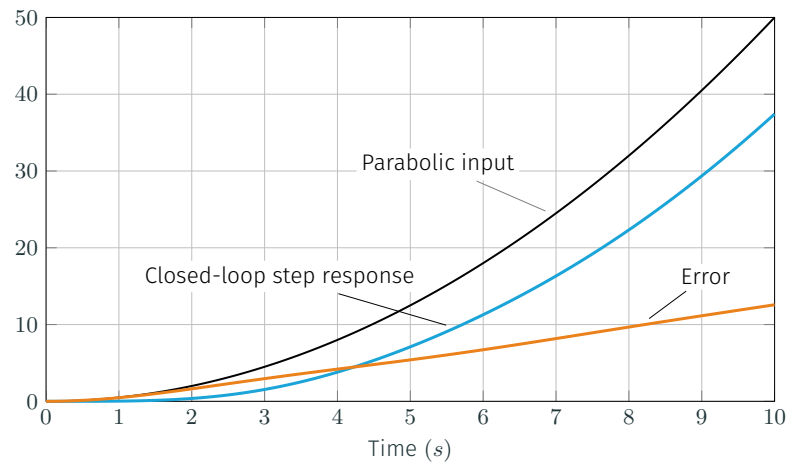
Type 1 System Response to a Parabolic Command



Infinite steady-state offset

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Type 1 System Response to a Parabolic Command



Infinite steady-state offset

23

Steady-State Errors by System Type

Type	$r(t) = 1$	$r(t) = t$	$r(t) = t^2$
0	$\frac{1}{1+\gamma}$	∞	∞
1	0	$\frac{1}{\gamma}$	∞
2	0	0	$\frac{1}{\gamma}$

Basic idea: Must have more integrators than the signal you're trying to track

Why not just add hundreds of integrators, and track anything?!

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Steady-State Errors by System Type - by the Book

The book uses a slightly different notation:

Type	$r(t) = 1$ Step (position)	$r(t) = t$ Ramp (velocity)	$r(t) = t^2$ Parabola (acceleration)
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

$$K_p = \lim_{s \rightarrow 0} K(s)G(s) \quad \text{Type 0}$$

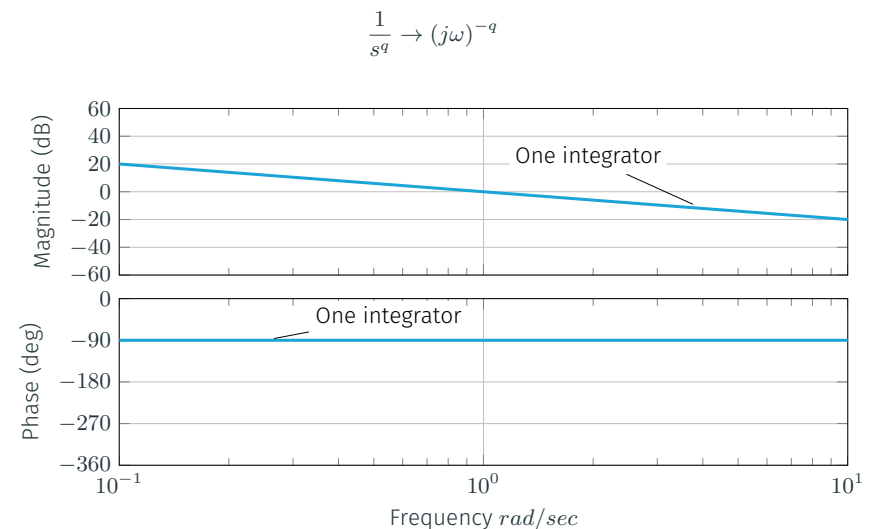
$$K_v = \lim_{s \rightarrow 0} sK(s)G(s) \quad \text{Type 1}$$

$$K_a = \lim_{s \rightarrow 0} s^2 K(s)G(s) \quad \text{Type 2}$$

The book differentiates constants between position, velocity and acceleration. We just use γ for all of them.

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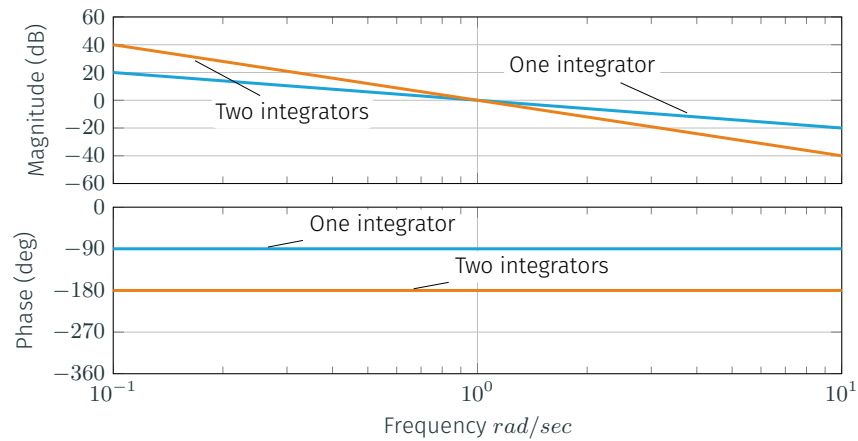
Negative Impact of Adding Integrators



26

Negative Impact of Adding Integrators

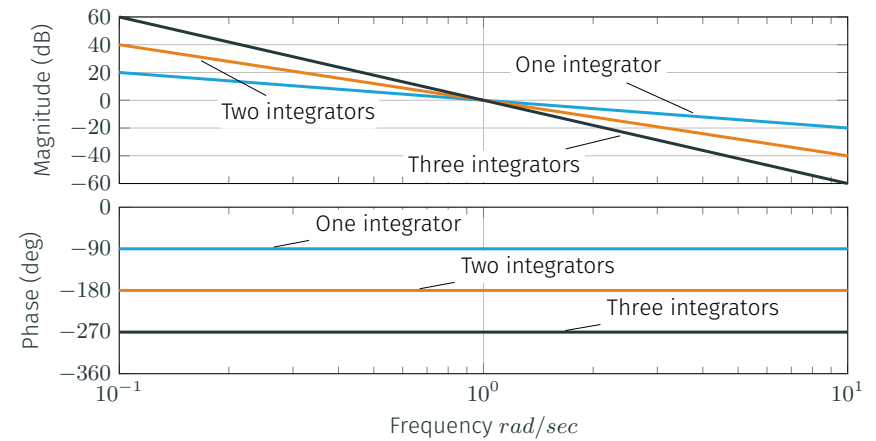
$$\frac{1}{s^q} \rightarrow (j\omega)^{-q}$$



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Negative Impact of Adding Integrators

$$\frac{1}{s^q} \rightarrow (j\omega)^{-q}$$

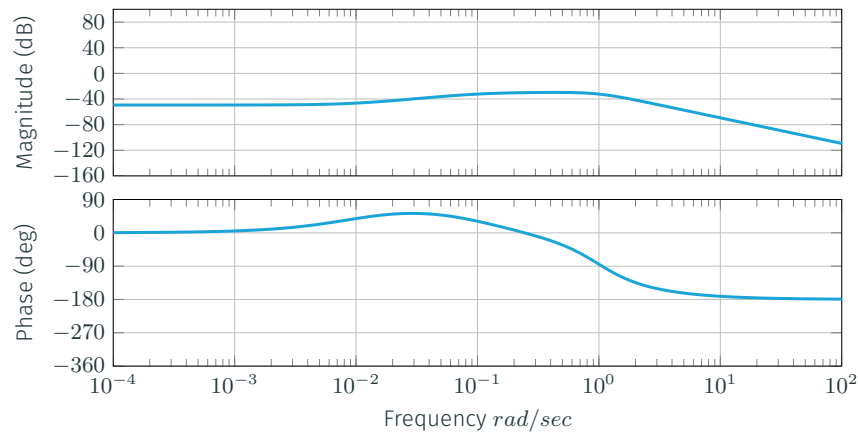


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Negative Impact of Adding Integrators

How do the two systems differ in terms of stability margins?

$$G(s) \quad \text{vs} \quad \frac{G(s)}{s^q}$$

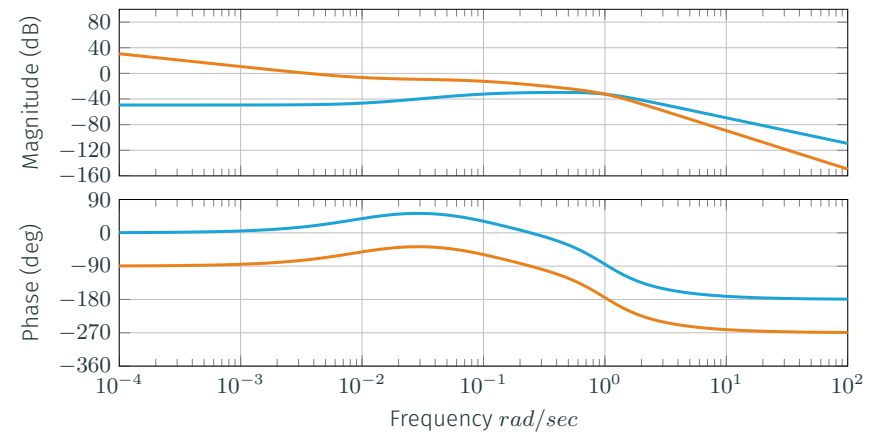


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Negative Impact of Adding Integrators

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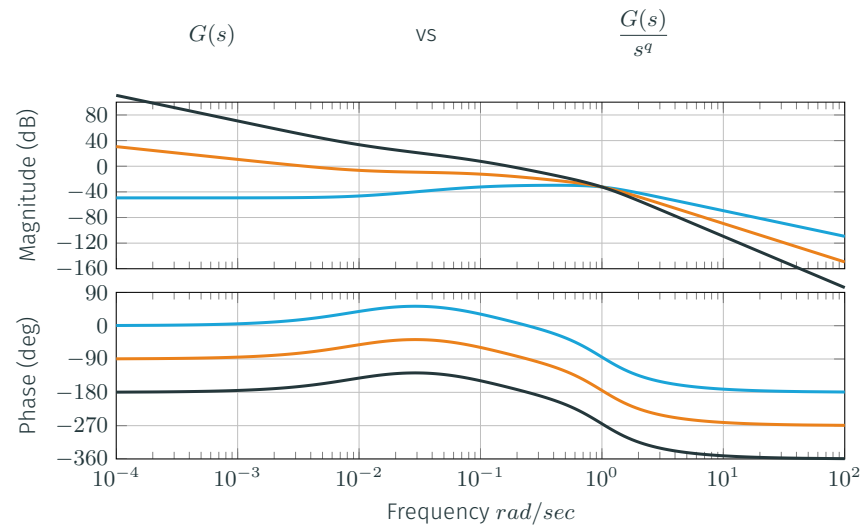
$$G(s) \quad \text{vs} \quad \frac{G(s)}{s^q}$$



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Negative Impact of Adding Integrators

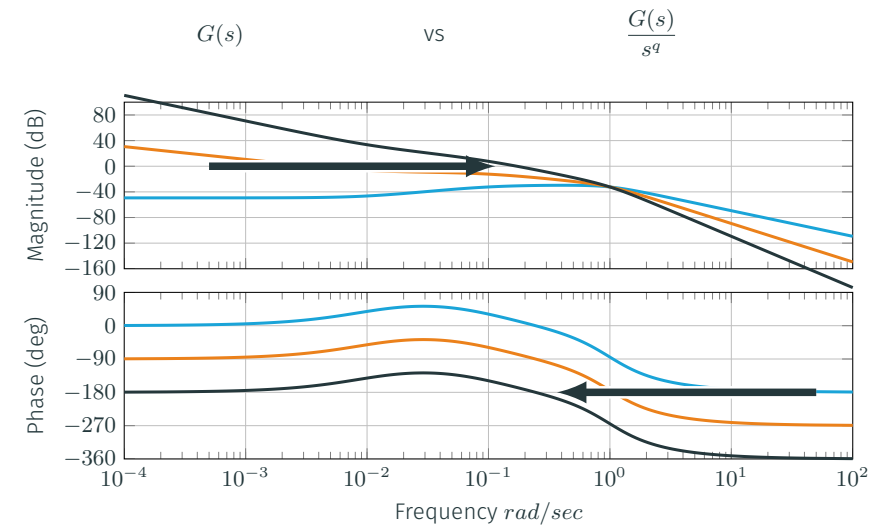
How do the two systems differ in terms of stability margins?



27

Negative Impact of Adding Integrators

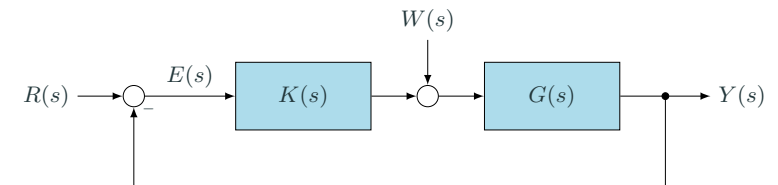
How do the two systems differ in terms of stability margins?



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Constant Disturbance Rejection

Constant Disturbance Rejection



System response with respect to a disturbance $w(t)$ is:

$$E(s) = \frac{G(s)}{1 + K(s)G(s)} W(s)$$

What are the conditions for zero steady-state offset with respect to a constant disturbance?

Constant Disturbance Rejection

Suppose the system has q integrators, and K has none

$$G(s) = \frac{B(s)}{s^q A(s)}$$

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Constant Disturbance Rejection

Suppose the system has q integrators, and K has none

$$G(s) = \frac{B(s)}{s^q A(s)}$$

The error is:

$$\begin{aligned} E(s) &= \frac{G(s)}{1 + K(s)G(s)} W(s) \\ &= \frac{B(s)}{s^q A(s) + K(s)B(s)} \cdot \frac{1}{s} \end{aligned}$$

Then the steady-state error is:

$$\lim_{s \rightarrow 0} sE(s) = \frac{1}{K(0)}$$

Integrators in the system do not reject disturbances!

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Integrators in the Controller

Suppose the controller K has r integrators

$$K(s) = \frac{S(s)}{s^r R(s)}$$

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Integrators in the Controller

Suppose the controller K has r integrators

$$K(s) = \frac{S(s)}{s^r R(s)}$$

The error is:

$$\begin{aligned} E(s) &= \frac{B(s)}{s^q A(s) + K(s)B(s)} \cdot \frac{1}{s} \\ &= s^r \frac{B(s)R(s)}{s^r s^q A(s)R(s) + S(s)B(s)} \cdot \frac{1}{s} \end{aligned}$$

The controller's integrators do reject the disturbance:

- One pole at 0 $\rightarrow$ rejects constant disturbance
- Two poles at 0 $\rightarrow$ rejects ramp disturbance
- $\vdots$
- etc

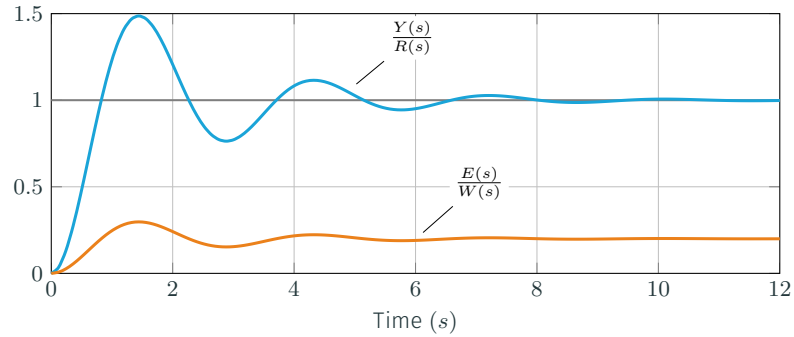
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Example

$$G(s) = \frac{1}{s(s+1)}$$

$$K(s) = 5$$

- System has an integrator
- Controller doesn't



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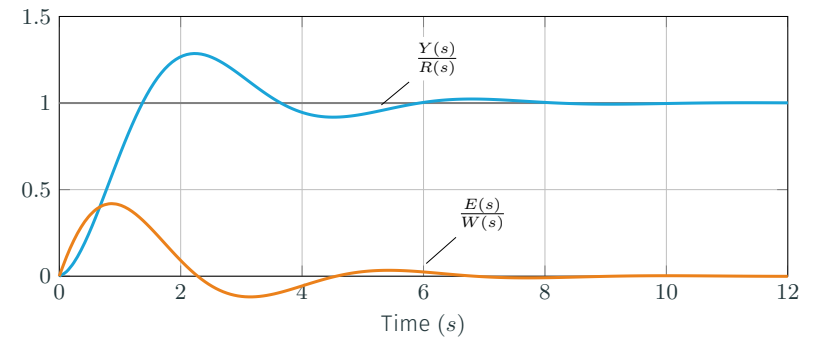
Waterbed Effect

Example

$$G(s) = \frac{1}{s+1}$$

$$K(s) = 0.1 + \frac{2.2}{s}$$

- System doesn't have an integrator
- Controller does (PI)



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Dynamic Disturbance Rejection

We want to reject 'complex' signals.

Consider a sinusoid

$$r(t) = \sin(\omega t) ,$$

or a mix of sinusoids.

The rejection of these signals, or the sensitivity to them, is given by the sensitivity function

$$\left| \frac{1}{1 + K(j\omega)G(j\omega)} \right| = |S(j\omega)|$$

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Limitations of Disturbance Rejection

Theorem Bode's Integral Formula

Assume we have a closed-loop stable system with open-loop unstable poles p_i , $i = 1, 2, \dots, P$ and a strictly proper open-loop transfer function $K(s)G(s)$. The sensitivity function satisfies the condition

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi \sum_{i=1}^P \operatorname{Re}(p_i)$$

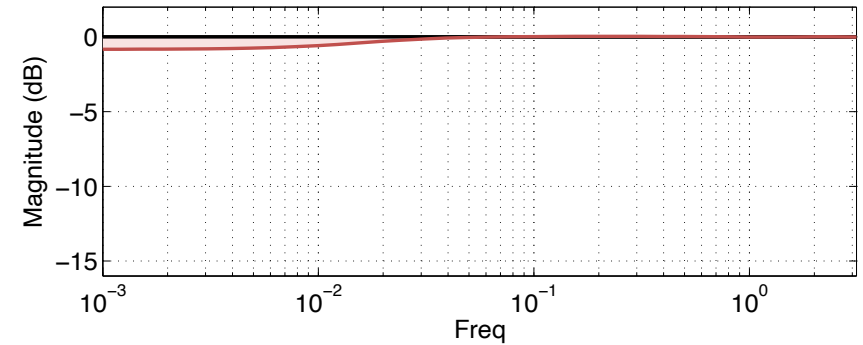
This is a fundamental limit on how well the system can perform:

- If we damp noise for some frequencies $|S(j\omega)| < 1$, then we must **amplify** it $|S(j\omega)| > 1$ at others!
- This is called the **waterbed effect**
- Harder to get good disturbance rejection behaviour out of unstable systems (those with many unstable poles)

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Example

$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



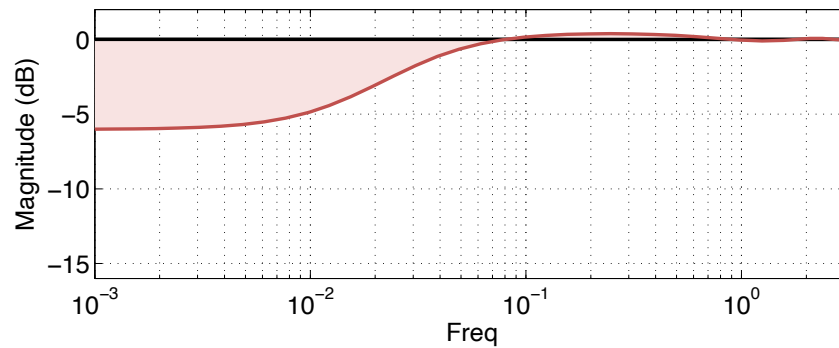
$$K_p = 0.1$$

$$\max_{\omega} S(j\omega) = 0.04dB$$

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Example

$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



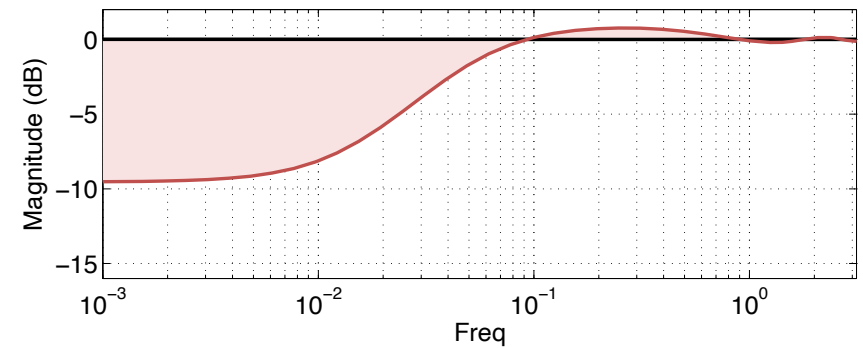
$$K_p = 1$$

$$\max_{\omega} S(j\omega) = 0.4dB$$

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Example

$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



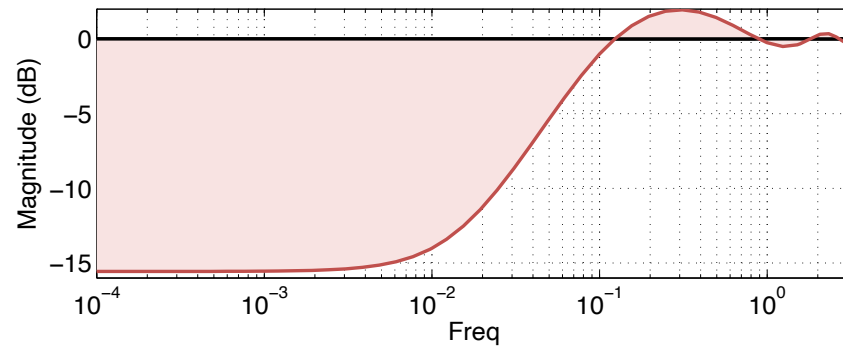
$$K_p = 2$$

$$\max_{\omega} S(j\omega) = 0.8dB$$

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Example

$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



$$K_p = 5$$

$$\max_{\omega} S(j\omega) = 2dB$$

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Summary

Robustness: The farther the 'nominal' Nyquist curve is from the -1 point, the more likely the real system will be stable.

"Margins" measure how far your system is from unstable

- Gain margin
- Phase margin
- Delay margin

Steady-state offset

- Need to have as many integrators in your controller as are in the signal to track / reject if you want zero steady-state error

Waterbed effect

- There is a fundamental limit to how well a controller can work
- Cannot improve noise rejection / tracking at all frequencies

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